MATH2040 Linear Algebra II

Tutorial 8

November 3, 2016

1 Examples:

Example 1

Let $V = \mathbb{C}^2$ with the inner product $\langle x, y \rangle = \sum_{i=1}^{2} x_i \overline{y_i}$, and $T(a_1, a_2) = (2a_1 + ia_2, (1-i)a_1)$ be a linear operator on V. If $x = (3-i, 1+2i)$, evaluate $T^*(x)$.

Solution

Since T^* satisfies $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ for all $x, y \in V$. So for any $v = (v_1, v_2)$

$$\begin{aligned} \langle (v_1, v_2), T^*(3-i, 1+2i) \rangle &= \langle T(v), (3-i, 1+2i) \rangle \\ &= \langle T(v), (3-i, 1+2i) \rangle \\ &= \langle (2v_1 + iv_2, (1-i)v_1), (3-i, 1+2i) \rangle \\ &= 5v_1 - iv_1 - v_2 + 3iv_2 \\ &= \langle (v_1, v_2), (5+i, -1-3i) \rangle \end{aligned}$$

So $T^*(3-i, 1+2i) = (5+i, -1-3i)$.

Example 2

Let T be a linear operator on an inner product space V. Prove that $||T(x)|| = ||x||$ for all $x \in V$ if and only if $\langle T(x), T(y) \rangle = \langle x, y \rangle$ for all $x, y \in V$.

Solution

" $\Rightarrow$ " Suppose that $||T(x)|| = ||x||$ for all $x \in V$.

Recall that in tutorial 6, we have shown that for any $x, y \in V$:

$$\langle x, y \rangle = \frac{1}{4} \sum_{k=1}^{4} i^k ||x + i^k y||^2$$

if $\mathbb{F} = \mathbb{C}$.

And if $\mathbb{F} = \mathbb{R}$, we have similar result:

$$\langle x, y \rangle = \frac{1}{4}(||x+y||^2 - ||x-y||^2)$$

Using the above relation for the complex version, we have

$$\begin{aligned}\langle T(x), T(y)\rangle &= \frac{1}{4}\sum_{k=1}^{4} i^k ||T(x) + i^k T(y)||^2 \\ &= \frac{1}{4}\sum_{k=1}^{4} i^k ||T(x + i^k y)||^2 \\ &= \frac{1}{4}\sum_{k=1}^{4} i^k ||x + i^k y||^2 \\ &= \langle x, y\rangle\end{aligned}$$

" $\Leftarrow$ " Suppose that $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$.

Substituting $y = x$, we have the desired result.

Example 3

Let V be an inner product space, and let $y, z \in V$. Define $T : V \to V$ by $T(x) = \langle x, y\rangle z$ for all $x \in V$. First prove that T is linear. Then find an explicit expression for T^*.

Solution

For any $x_1, x_2 \in V$ and $c \in \mathbb{F}$,

$$\begin{aligned}T(cx_1 + x_2) &= \langle cx_1 + x_2, y\rangle z \\ &= \langle cx_1, y\rangle z + \langle x_2, y\rangle z \\ &= c\langle x_1, y\rangle z + \langle x_2, y\rangle z \\ &= cT(x_1) + T(x_2)\end{aligned}$$

So T is linear.

Since T^* satisfies the relation $\langle T(x), v\rangle = \langle x, T^*(v)\rangle$ for any $v \in V$, we consider

$$\begin{aligned}\langle x, T^*(v)\rangle &= \langle T(x), v\rangle \\ &= \langle \langle x, y\rangle z, v\rangle \\ &= \langle x, y\rangle \langle z, v\rangle \\ &= \langle x, \overline{\langle z, v\rangle} y\rangle \\ &= \langle x, \langle v, z\rangle y\rangle\end{aligned}$$

Therefore, $T^*(v) = \langle v, z\rangle y$ for any $v \in V$.

Example 4

For a linear operator T on an inner product space V, prove that if $T^*T = T_0$(which is the zero transformation), then $T = T_0$. Is the same result still true if the assumption is changed to $TT^* = T_0$ instead of $T^*T = T_0$?

Solution

If we want to show $T = T_0$, then we need to show $T(x) = T_0(x) = 0$ for all $x \in V$. Note for any $x \in V$

$$0 = \langle x, 0\rangle = \langle x, T^*T(x)\rangle = \langle T(x), T(x)\rangle = ||T(x)||^2$$

which implies that $T(x) = 0$ for all $x \in V$.

If the assumption is changed to $TT^* = T_0$, we can have the same result since

$$0 = \langle x, 0\rangle = \langle x, TT^*(x)\rangle = \langle T^*(x), T^*(x)\rangle = ||T^*(x)||^2$$

which means $T^* = T_0$.

Therefore, $T = T^{**} = T_0^* = T_0$.

2 Exercises:

<u>**Question 1**</u> (Section 6.3 Q3(c)):

Let $V = P_1(\mathbb{R})$ with the inner product $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)\, dt$, and $T(f) = f' + 3f$ be a linear operator on V. If $h(t) = 4 - 2t$, evaluate $T^*(h)$.

<u>**Question 2**</u> (Section 6.3 Q8):

Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Prove that if T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

<u>**Question 3**</u> (Section 6.3 Q9):

Let W be a finite-dimensional subspace of an inner product space V. Prove that if $V = W \oplus W^\perp$ and T is the projection on W along $W^\perp$, then $T = T^*$. (Hint: $N(T) = W^\perp$)

<u>**Question 4**</u> (Section 6.3 Q12):

Let V be an inner product space, and let T be a linear operator on V. Prove the following results:

(a) $R(T^*)^\perp = N(T)$.

(b) If V is finite-dimensional , then $R(T^*) = N(T)^\perp$. (Hint: You may first show $W = (W^\perp)^\perp$, where W is a finite-dimensional subspace of V)

Answers:

Question 1:

<u>Method 1</u> Since $T^*(h) \in V$, let $T^*(h) = at + b$. As $\langle f, T^*(h)\rangle = \langle T(f), h\rangle$ for all $f \in V$, we consider $f(t) = 1$ and $f(t) = t$:

$$\begin{cases} 2b = \langle 1, T^*(h)\rangle = \langle T(1), h\rangle = \langle 3, 4-2t\rangle = 24 \\ \dfrac{2}{3}a = \langle t, T^*(h)\rangle = \langle T(t), h\rangle = \langle 1+3t, 4-2t\rangle = 4 \end{cases}$$

Solving the above linear system, we have $a = 6, b = 12$, so $T^*(h) = 6t + 12$.

<u>Method 2</u> Let $\beta = \{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}t\}$ be an orthonormal basis of V. Then

$$[T]_\beta = \begin{pmatrix} 3 & \sqrt{3} \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad [T^*]_\beta = [T]_\beta^* = \begin{pmatrix} 3 & 0 \\ \sqrt{3} & 3 \end{pmatrix}$$

Then, $T^*(\frac{1}{\sqrt{2}}) = \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}t$ and $T^*(\frac{\sqrt{3}}{\sqrt{2}}t) = 3\frac{\sqrt{3}}{\sqrt{2}}t$.

So $T^*(h) = 6t + 12$.

Question 2:

Since T is invertible , so T^{-1} is well-defined. Note

$$(T^*)(T^{-1})^* = (T^{-1}T)^* = I^* = I \quad \text{and} \quad (T^{-1})^*(T^*) = (TT^{-1})^* = I^* = I$$

Question 3:

For any $x, y \in V$, $x = x_1 + x_2$ and $y = y_1 + y_2$, where $x_1, y_1 \in W$ and $x_2, y_2 \in W^\perp$.

Since

$$\langle x, T(y)\rangle = \langle x, y_1\rangle = \langle x_1, y_1\rangle + \langle x_2, y_1\rangle = \langle x_1, y_1\rangle$$

and

$$\langle x, T^*(y)\rangle = \langle T(x), y\rangle = \langle x_1, y_1\rangle + \langle x_1, y_2\rangle = \langle x_1, y_1\rangle = \langle x, T(y)\rangle.$$

Therefore, $T = T^*$.

Question 4:

(a) On one hand, suppose $x \in R(T^*)^\perp$. For any $y \in V$,

$$0 = \langle x, T^*(y) \rangle = \langle T(x), y \rangle.$$

So $T(x) = 0$ and $x \in N(T)$.

On the other hand, suppose $x \in N(T)$. For any $y \in V$,

$$0 = \langle 0, y \rangle = \langle T(x), y \rangle = \langle x, T^*(y) \rangle.$$

So $x \in R(T^*)^\perp$.

(b) Since V is finite-dimensional, applying the relation $W = (W^\perp)^\perp$ on (a), the result follows.